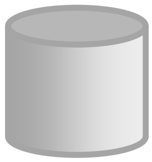
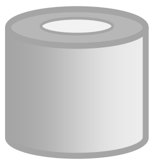
THE WAVE EQUATION Cylindrical Coordinates



$[u]_S = f$



$[u]_S = f$

$$\nabla^2 u + F(r,\theta,z) = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$[u]_S = f \qquad t > 0$ *boundary conditions*

$[u]_{t=t_0} = u_0$

initial conditions

$\left[\frac{\partial}{\partial t}u\right]_{t=t_0} = u_0$

solid cylinder

$(r,\theta,z) \in [0,r_1)\times[-\pi,\pi]\times(0,L) \subset \mathbb{R}^3$

hollow cylinder

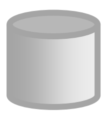
$(r,\theta,z) \in (r_1,r_2)\times[-\pi,\pi]\times(0,L) \subset \mathbb{R}^3$

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

STEADY STATE PROBLEM - PE

$$\nabla^2 u_s + F(r,\theta,z) = 0$$

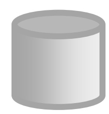
$[u_s]_S = f$



TRANSIENT PROBLEM - HE

$$\nabla^2 U = \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}$$

$[U]_S = 0$



initial conditions

$[U]_{t=t_0} = u_0 - u_s$

$\left[\frac{\partial}{\partial t}U\right]_{t=t_0} = u_1$

supplemental eigenvalue problems

$\Theta'' = \eta\Theta \quad \overset{SLP}{\Rightarrow} \quad \Theta_n'' = -n^2\Theta_n \qquad n = 0,1,2,\ldots$

$\Theta(\theta+2\pi) = \Theta(\theta)$

$\eta_0 = 0 \qquad \Theta_0(\theta) = 1$

$\eta_n = -n^2 \qquad \Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$

$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = \mu R \qquad R_{nm}'' + \frac{1}{r}R_{nm}' - \frac{n^2}{r^2}R_{nm} = -\lambda_{nm}^2 R$

$R(0) < \infty \quad \overset{SLP}{\Rightarrow} \quad \mu_{nm} = -\lambda_{nm}^2$

$R(r_1) = 0 \qquad r^2R_{nm}'' + rR_{nm}' + (r^2\lambda_{nm}^2 - n^2)R_{nm} = 0$

$R_{nm}(r) = J_n(\lambda_{nm} r) \qquad n = 0,1,2,\ldots \quad m = (0),1,2,\ldots$

$Z'' = \gamma Z \qquad Z_k'' = -\omega_k^2 Z_k$

$[Z]_{z=0} = 0 \quad \overset{SLP}{\Rightarrow} \quad \gamma_k = -\omega_k^2$

$[Z]_{z=K} = 0 \qquad Z_k(z)$

SEPARATION OF VARIABLES

$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t) \qquad \frac{\nabla^2\Phi}{\Phi} = \frac{1}{\alpha}\frac{T''}{T} = \beta$$

HELMHOLTZ EQUATION

$\nabla^2\Phi = \beta\Phi$

$\Phi(r,\theta,z) = R(r)\Theta(\theta)Z(z)$

$R_{nm},\ \Theta_n,\ Z_k$

$\beta_{nmk} = -(\lambda_{nm}^2 + \omega_k^2)$

$\frac{1}{v^2}\frac{T''}{T} = \beta$

$T_{nmk}(t) = c_1\cos(v^2\beta_{nmk}t) + c_2\sin(v^2\beta_{nmk}t)$

STEADY STATE SOLUTION

TRANSIENT SOLUTION

$$U(r,\theta,z,t) = \Phi(r,\theta,z)T(t)$$

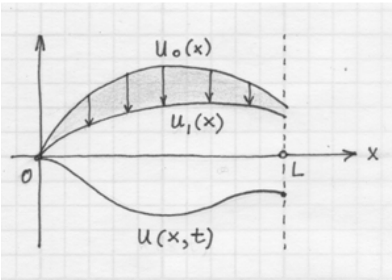
see p.654 for the case of solid cylinder, and p.658 for the case of hollow cylinder

SOLUTION OF IBVP

$$u(r,\theta,z,t) = u_s(r,\theta,z) + U(r,\theta,z,t)$$

VIII.3.5 THE WAVE EQUATION

VIII.3.5.1 1-D Cartesian BASIC homogeneous equation with homogeneous boundary conditions



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad u(x,t),\ \ x\in(0,L),\ t>0$$

Initial conditions: $u(x,0)=u_0(x)$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x)$$

Boundary conditions: $u(0,t)=0,\qquad t>0 \qquad$ (I)

$$k_2\frac{\partial u(L,t)}{\partial x} + h_2 u(x,t) = 0,\ \ t>0 \qquad \text{(III)}$$

Denote $H_2 = \dfrac{h_2}{k_2} > 0$

1. Separation of variables we assume that the function $u(x,t)$ can be represented as a product of two functions each of a single variable

$u(x,y) = X(x)\,T(t)$ substitute into equation

$a^2 X''(x)\,T(t) = X(x)\,T''(t)$ after separation of variables, one gets

$\dfrac{X''}{X} = \dfrac{1}{a^2}\dfrac{T''}{T} = \mu$ with a separation constant μ

boundary conditions:

$\underline{x=0}$ $\quad X(0)\,T(t)=0 \quad\Rightarrow\quad X(0)=0$

$\underline{x=L}$ $\quad X'(L)\,T(t) + H_2 X(L)\,T(t) = 0 \quad\Rightarrow\quad X'(L)+H_2X(L)=0$

2. Sturm-Liouville problem $X'' - \mu X = 0$

This Sturm-Liouville problem has solution with $\mu_n = -\lambda_n^2$:

eigenvalues λ_n are positive roots of equation $\lambda\cos\lambda L + H_2\sin\lambda_n L = 0$

eigenfunctions $X_n(x) = \sin\lambda_n x$

Then solutions of the second differential equation $T'' + \lambda_n^2 a^2 T = 0$ are

$T_n(t) = c_1\cos\lambda_n at + c_2\sin\lambda_n at$

Solution: $u_n(x,t) = X_nT_n = \sin(\lambda_n x)(c_1\cos\lambda_n at + c_2\sin\lambda_n at)$

Then solution of the wave equation is a superposition

$$u(x,t) = \sum_{n=1}^{\infty}\sin(\lambda_n x)(b_n\cos\lambda_n at + d_n\sin\lambda_n at)$$

initial conditions: $\underline{t=0}$ $$u(x,0)=\sum_{n=1}^{\infty} b_n \sin(\lambda_n x)=u_0(x)$$

which is a generalized Fourier series expansion of the function $f(x)$ over the interval $(0,L)$ with coefficients

$$b_n=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_0(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

The derivative with respect to *t* of the assumed solution is

$$\frac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}\lambda_n a\sin\lambda_n x\left(-b_n\sin\lambda_n at+d_2\cos\lambda_n at\right)$$

Then the second initial condition yields

$\underline{t=0}$ $$\frac{\partial u(x,0)}{\partial t}=\sum_{n=1}^{\infty} d_n\lambda_n a\sin\lambda_n x=u_1(x)$$

It can be treated as a Fourier series with coefficients

$$d_n\lambda_n a=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\int_0^L \sin^2\lambda_n x dx}=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}}$$

then

$$d_n=\frac{\int_0^L u_1(x)\sin\lambda_n x dx}{\lambda_n a\left(\frac{L}{2}-\frac{\sin 2\lambda_n L}{4\lambda_n}\right)}$$

Then the solution of the initial-boundary value problem is:

3. Solution $$u(x,t)=\sum_{n=1}^{\infty}\left[b_n\cos(\lambda_n at)+d_n\sin(\lambda_n at)\right]\sin(\lambda_n x)$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{\sin(\lambda_n x)}{\left(\frac{L}{2}-\frac{\sin(2\lambda_n L)}{4\lambda_n}\right)}\left\{\left[\int_0^L u_0(x)\sin(\lambda_n x)dx\right]\cos\lambda_n at+\left[\frac{\int_0^L u_1(x)\sin(\lambda_n x)dx}{\lambda_n a}\right]\sin(\lambda_n at)\right\}$$

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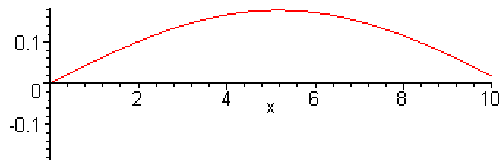
4. Normal modes of string vibration

The solution of the Wave Equation is obtained as a sum of terms

$$u_n(x,t) = X_n T_n = sin(\lambda_n x)(c_1 \cos \lambda_n at + c_2 \sin \lambda_n at)$$

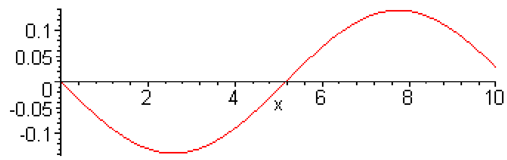
which we call the basic solutions. However, in the context of contributions to the vibration of a string, these functions are known as **normal modes**.
In our example, for $n = 1,2,3,4,...$, they have the following shapes (see the Maple the file for animation):

```
> m1:=subs(n=1,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m1},x=0..L,t=0..9);
```



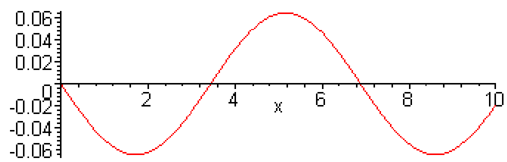
fundamental mode

```
> m2:=subs(n=2,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m2},x=0..L,t=0..9);
```



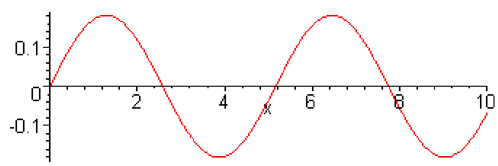
first overtone

```
> m3:=subs(n=3,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m3},x=0..L,t=0..9);
```



second overtone

```
> m4:=subs(n=4,X[n]*(b[n]*cos(lambda[n]*a*t)+d[n]*sin(lambda[n]*a*t))):
> animate({m4},x=0..L,t=0..9);
```



third overtone

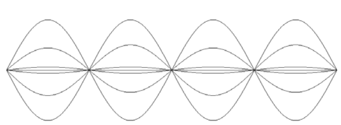
overtones

The first of these normal modes is called the **fundamental mode**, while the others are referred to as the **first overtone**, the **second overtone**, and so on. The **frequency** of oscillation of the normal mode increases with its number and is determined by the corresponding eigenvalue λ_n and coefficient a, which has a physical meaning related to the speed of wave propagation (speed of sound). Fixed points exist in the vibration of overtones.

The entire motion of the string is a superposition of vibration of all overtones with a different amplitude. The participation of different modes in the string's vibration is determined by the initial conditions.

If representing the initial shape of the string at rest requires the use of different modes, then all of them will be present in the undamped vibration of the string.

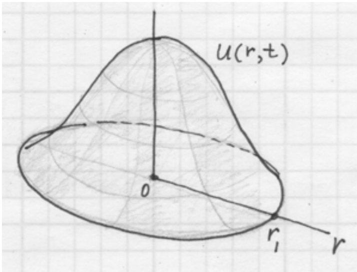
standing waves



However, if the initial shape of the string exactly matches one of the overtones, then only that mode will be present in the string's vibration.

This phenomenon is known as **standing waves**. Standing waves do not propagate; they only oscillate, maintaining the same shape.

VIII.3.5.2 1-D polar coordinates *Wave Equation in polar coordinates with angular symmetry*



$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

$u(r,t),\ 0\le r<r_1,\ t>0$

Initial conditions: $u(r,0)=u_0(r)$

$\frac{\partial u(r,0)}{\partial t}=u_1(r)$

Boundary condition: $u(r_1,t)=0$ $t>0$ (Dirichlet)

$u(0,t)<\infty$

1. Separation of variables

Assume

$u(r,y)=R(r)\,T(t)$

Substitute into the equation

$$R''T+\frac{1}{r}RT'=\frac{1}{v^2}RT''$$

After separation of variables (division by RT), we receive

$$\frac{R''}{R}+\frac{1}{r}\frac{R}{R}=\frac{1}{v^2}\frac{T''}{T}=\mu$$ with a separation constant μ .

boundary condition $r=r_1$ $u(r_1,t)=R(r_1)\,T(t)=0 \Rightarrow R(r_1)=0$

2. Solution of Sturm-Liouville problem

Consider the equation for $R(r)$ for which we have a homogeneous boundary condition:

$$R''+\frac{1}{r}R'-\mu R=0 \qquad R(r_1)=0$$

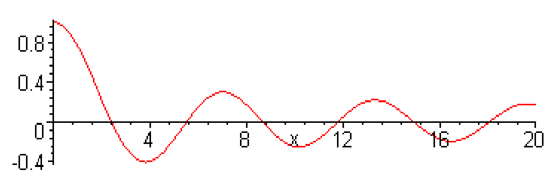
That is the Eigenvalue problem for the Bessel equation of 0^{th} order, solution for which is presented in VII.2, p.509.

Separation constant $\mu_n=-\lambda_n^2$

Eigenfunctions $R_n(r)=J_0(\lambda_n r)$

Eigenvalues are the roots of $J_0(\lambda_n r_1)=0$

The figure shows the graph of the function $w(\lambda)=J_0(\lambda r_1)$ with $r_1=1$



The weight function $p(r)=r$

Orthogonality $\int_0^{r_1} J_0(\lambda_n r)J_0(\lambda_m r)\,r\,dr=0$ for $n\ne m$

Norm $\left\|R_n(r)\right\|_p^2=r_1^2J_1^2(\lambda_n r_1)/2$

solution for T

The result of a negative separation constant $\mu=-\lambda^2$ agrees with a physical sense of solution for $T(t)$. Equation for T

$$\frac{1}{a^2}\frac{T''}{T}=\mu=-\lambda_n^2$$

Then solutions $T_n(t)$ with determined eigenvalues are

$$T_n(t)=a_n\cos(v\lambda_n t)+b_n\sin(v\lambda_n t)$$

3. Solution

$$u(r,t)=\sum_{n=1}\left[a_n\cos(v\lambda_n t)+b_n\sin(v\lambda_n t)\right]J_0(\lambda_n r)$$

We will choose the values of coefficients in such a way that initial conditions are satisfied.

4. Initial conditions

Consider the first initial condition

$$u(r,0)=\sum_{n=1}^{\infty}a_n J_0(\lambda_n r)=u_0(r)$$

then coefficients for the generalized Fourier series are defined as

$$a_n=\frac{\int_0^{r_1}u_0(r)J_0(\lambda_n r)r\,dr}{\int_0^{r_1}J_0^2(\lambda_n r)r\,dr}$$

$$a_n=\frac{\int_0^{r_1}ru_0(r)J_0(\lambda_n r)dr}{r_1^2J_1^2(\lambda_n r_1)/2}$$

The second condition for the derivative with respect to time

$$\frac{\partial u(r,t)}{\partial t}=\sum_{n=1}^{\infty}J_0(\lambda_n r)\left(-a_n\lambda_n v\sin\lambda_n vt+b_n\lambda_n v\cos\lambda_n vt\right)$$

becomes

$$\frac{\partial u(r,0)}{\partial t}=\sum_{n=1}^{\infty}b_n\lambda_n v\,J_0(\lambda_n r)=u_1(r)$$

Then coefficients in this generalized Fourier expansion are

$$b_n\lambda_n v=\frac{\int_0^{r_1}u_1(r)J_0(\lambda_n r)r\,dr}{\int_0^{r_1}rJ_0^2(\lambda_n r)r\,dr}\quad\Rightarrow\quad b_n=\frac{\int_0^{r_1}ru_1(r)J_0(\lambda_n r)dr}{v\lambda_n r_1^2J_1^2(\lambda_n r_1)/2}$$

Then solution of the initial-boundary value problem is

5. Solution

$$u(r,t)=\sum_{n=1}^{\infty}\left[a_n\cos(\lambda_n vt)+b_n\sin(\lambda_n vt)\right]J_0(\lambda_n r)$$

$$u(r,t)=\frac{r_1^2}{2}\sum_{n=1}^{\infty}\left\{\left[\int_0^{r_1}u_0(r)J_0(\lambda_n r)r\,dr\right]\cos(\lambda_n vt)+\left[\frac{1}{v\lambda_n}\int_0^{r_1}u_1(r)J_0(\lambda_n r)r\,dr\right]\sin(\lambda_n vt)\right\}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n r_1)}$$

VIII.3.6 SINGULAR STURM_LIOUVILLE PROBLEM – CIRCULAR STRING

We studied a regular Sturm-Liouville Problem in which the ordinary differential equation is set in the finite interval and both boundary conditions do not vanish. In a singular Sturm-Liouville problem not all of these conditions hold. Usually, the interval is not finite, and one or both boundary conditions are missing. Instead of boundary conditions, when the solution may not exist at the boundaries, the eigenfunctions should satisfy some limiting conditions. One of such requirements can be the following:
Let y_1 and y_2 be eigenfunctions corresponding to two distinct eigenvalues λ_1 and λ_2, correspondingly. Then they have to satisfy the following condition:

$$\lim_{x\to x_2^-} p(x)\left[y_1(x)y_2'(x)-y_2(x)y_1'(x)\right] = \lim_{x\to x_1^+} p(x)\left[y_1(x)y_2'(x)-y_2(x)y_1'(x)\right]$$

In the other cases the absence of boundary conditions is because of the periodical or cycled domain, when we demand that the solution should be continuous and smooth
$y(x_1)=y(x_2)$ and $y'(x_1)=y'(x_2)$
In this case, it is still possible to have the orthogonal set of solutions $\{y_n(x)\}$ on $[x_1,x_2]$.
We will not study the formal approach to solution of such problems, but rather discuss the practical examples of its application.
Here, we consider an interesting example of a singular SLP in a cycled domain with no boundary conditions. Physical demonstration of this example can be seen on the ceiling of the hall of the Eyring Science Building.

Example 1 Consider vibration of a thin closed ring string of radius r described in polar coordinates by deflection over the plane $z=0$
$u(\theta,t),\ \theta\in[0,2\pi]\ ,\ t>0$
The Wave Equation reduces to

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=a^2\frac{\partial^2 u}{\partial t^2} \qquad r=const$$

with initial conditions

$$u(\theta,0)=u_0(\theta)$$

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)$$

There are no boundaries for a closed string, but rather a physical condition for a continuous and smooth string:

$$u(0,t)=u(2\pi,t) \quad t>0$$

$$\frac{\partial u}{\partial\theta}(0,t)=\frac{\partial u}{\partial\theta}(2\pi,t) \quad t>0$$

separation of variables

Assume $u(\theta,t)=\Theta(\theta)\ T(t)$

Substitute into equation $\frac{1}{r^2}\Theta''T=a^2\ \Theta\ T''$

Separate variables $\frac{\Theta''}{\Theta}=a^2r^2\frac{T''}{T}=\mu$ $\quad\mu$ is a separation constant

Consider

$$\frac{\Theta''}{\Theta}=\mu$$

$$\Theta''-\mu\Theta=0$$

We already have experience with solution of this special equation for regular Sturm-Liouville Problems and know that in all cases except the case of both boundary conditions of Neumann type, only a negative separation constant ,

$\mu = -\lambda^2$, generates eigenvalues and eigenfunctions. General solution in this case is

$$\Theta(\theta) = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$$

This solution suits our problem because it is periodic. The values of λ which satisfy periodicity on the interval $\theta \in [0,2\pi]$, are

$$\lambda_n = \frac{2n\pi}{2\pi} = n$$

Therefore, solutions are

$$\Theta_n(\theta) = c_{1,n} \cos n\theta + c_{2,n} \sin n\theta$$

Obviously, that for all $n = 0,1,2,...$ 2π is a period for this solution and for its derivative

$$\Theta_n'(\theta) = -c_{1,n} n \sin n\theta + c_{2,n} n \cos n\theta$$

With these values of the separation constant, $\mu_n = -\lambda_n^2 = -n^2$, $n = 0,1,2,...$ consider the equation for $T(t)$:

$$a^2 r^2 \frac{T''}{T} = -n^2$$

$$T'' + \frac{n^2}{a^2 r^2} T = 0$$

which also has a periodic (in t) general solution

$$T_n(t) = c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t$$

Then periodic solution of the wave equation can be constructed in the form of an infinite series:

$$u(\theta,t) = \Theta(\theta)T(t) = \sum_{n=0}^{\infty} \Theta_n(\theta) T_n(t)$$

$$= \sum_{n=0}^{\infty} \left(c_{1,n} \cos n\theta + c_{2,n} \sin n\theta\right)\left(c_{3,n} \cos \frac{n}{ar} t + c_{4,n} \sin \frac{n}{ar} t\right)$$

$$= \sum_{n=0}^{\infty} \left(c_{1,n}c_{3,n} \cos n\theta \cos \frac{n}{ar} t + c_{1,n}c_{4,n} \cos n\theta \sin \frac{n}{ar} t + c_{2,n}c_{3,n} \sin n\theta \cos \frac{n}{ar} t + c_{2,n}c_{4,n} \sin n\theta \sin \frac{n}{ar} t\right)$$

$$= \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta \cos \frac{n}{ar} t + b_{2,n} \cos n\theta \sin \frac{n}{ar} t + b_{3,n} \sin n\theta \cos \frac{n}{ar} t + b_{4,n} \sin n\theta \sin \frac{n}{ar} t\right)$$

where coefficients b are new arbitrary constants which can be chosen in such a way that this solution will satisfy the initial conditions.
Consider the first initial condition:

$$t = 0 \qquad u(\theta,0) = u_0(\theta) = \sum_{n=0}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right)$$

$$= b_{1,0} + \sum_{n=1}^{\infty} \left(b_{1,n} \cos n\theta + b_{3,n} \sin n\theta\right)$$

which can be treated as a standard Fourier series expansion of the function $u_0(\theta)$ on the interval $[0,2\pi]$. Therefore, the coefficients of this expansion are

$$b_{1,0} = \frac{1}{2\pi} \int_0^{2\pi} u_0(\theta) d\theta$$

$$b_{1,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \cos n\theta d\theta$$

$$b_{3,n} = \frac{1}{\pi} \int_0^{2\pi} u_0(\theta) \sin n\theta d\theta$$

For the second initial condition, differentiate the solution first with respect to t

$$\frac{\partial u}{\partial t}(\theta,t)=\sum_{n=0}^{\infty}\left(-b_{1,n}\frac{n}{ar}\cos n\theta\sin\frac{n}{ar}t+b_{2,n}\frac{n}{ar}\cos n\theta\cos\frac{n}{ar}t-b_{3,n}\frac{n}{ar}\sin n\theta\sin\frac{n}{ar}t+b_{4,n}\frac{n}{ar}\sin n\theta\cos\frac{n}{ar}t\right)$$

then apply the second initial condition

$$\frac{\partial u}{\partial t}(\theta,0)=u_1(\theta)\qquad =\sum_{n=0}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

$$=b_{2,n}\cdot 0+\sum_{n=1}^{\infty}\left(b_{2,n}\frac{n}{ar}\cos n\theta+b_{4,n}\frac{n}{ar}\sin n\theta\right)$$

Where the coefficients are determined as

$$b_{2,0}\cdot 0=\frac{1}{2\pi}\int_0^{2\pi}u_1(\theta)d\theta$$

$$b_{2,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\cos n\theta d\theta$$

$$b_{4,n}=\frac{ar}{n}\frac{1}{\pi}\int_0^{2\pi}u_1(\theta)\sin n\theta d\theta$$

Coefficient $b_{2,0}$ can be any constant, it will not influence the initial speed of the string, but not to influence the initial shape of the string it has to be chosen equal to zero (otherwise, initially the string will shifted by $b_{2,0}$ and will not be centered over the plane $z=0$):

$b_{2,0}=0$

Therefore, solution of the problem is given by the infinite series

$$u(\theta,t)=b_{1,0}+\sum_{n=1}^{\infty}\left(b_{1,n}\cos n\theta\cos\frac{n}{ar}t+b_{2,n}\cos n\theta\sin\frac{n}{ar}t+b_{3,n}\sin n\theta\cos\frac{n}{ar}t+b_{4,n}\sin n\theta\sin\frac{n}{ar}t\right)$$

where coefficients are determined according to abovementioned formulas.

Consider particular cases (Maple examples):

1) isolated wave

2) standing waves